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# Network Optimization Models for Resource Allocation in Developing Military Countermeasures

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A military arms race is characterized by an iterative development of measures and countermeasures. An attacker attempts to introduce new weapons in order to gain some advantage, whereas a defender attempts to develop countermeasures that can mitigate or even eliminate the effects of the weapons. This paper addresses the defender's decision problem: given limited resources, which countermeasures should be developed and how much should be invested in their development to minimize the damage caused by the attacker's weapons over a certain time horizon. We formulate several optimization models, corresponding to different operational settings, as constrained shortest-path problems and variants thereof. We then demonstrate the potential applicability and robustness of this approach with respect to various scenarios.

*Subject classifications:* arms race; network optimization; constrained shortest path.

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## 1. Introduction

The term *arms race* is typically used to describe military buildup efforts by countries that are in conflict with one another (e.g., US versus the USSR during the Cold War era, India versus Pakistan, Greece versus Turkey, etc.). Studies of such phenomena typically address *strategic* issues and are commonly found in the political, economics, and strategic planning literature—see, for example, the recent study of military technology races in Koubi (1999). In contrast, the present paper addresses *operational* aspects that arise in arms races. In particular, it focuses on how much resources should be invested by a defender in such a race and how to time these investments. We consider an arms race between two asymmetric parties: Red ( $R$ ) and Blue ( $B$ ).  $R$  is the *attacker*, who is trying to develop an assortment of new weapons to attack the *defender*  $B$ . Being aware of  $R$ 's capabilities, intentions, and activities,  $B$  is trying to develop countermeasures (CMs) that will mitigate, or even neutralize, the effects of  $R$ 's weapons. The CMs may be technological, tactical, or both.

If  $R$  completes the development of a certain weapon and makes it operational before  $B$  is ready with appropriate CMs, then  $R$  inflicts a certain damage on  $B$  (typically measured in casualties and economic damages) per each time-unit until an appropriate CM becomes operational.

If  $B$  wins the race and a CM is operational before  $R$  deploys a weapon, then the damage to  $B$  is smaller when that weapon becomes available. If  $B$ 's CMs are perfectly effective against that weapon (see Etcheson 1989), the damage to  $B$  can be as low as zero. Given a set of existing and potential weapons to be deployed by  $R$ , the problem that  $B$  faces is how to utilize its limited resources to develop the most effective mix of CMs—a mix that minimizes total damage.

An example of the settings addressed in this paper is the counterinsurgency warfare faced by coalition forces in Iraq and Afghanistan (2003–2009) where the insurgents develop and deploy new types of improvised explosive devices (IED), with ever-increasing lethal capability, while the coalition forces continue to develop technologies, tactics, techniques, and procedures to respond to that threat (see, e.g., Neuneck 2008).

Arms race problems are related to a broader class of problems addressing investment rates in R&D projects that are carried out in competitive market environments (see, e.g., Golany and Rothblum 2008, Kamien and Schwartz 1971, and Spector and Zuckerman 1997). Most of the articles that have appeared in this literature have assumed the “winner-takes-all” hypothesis whereby the first party that achieves an advantage maintains it indefinitely and all other parties lose.

In a recent paper, Golany et al. (2012) analyze a stochastic version of the arms race problem of the kind described above. In contrast with the common “winner-takes-all” assumption, the models presented in Golany et al. (2012) address situations in which any advantage gained by one of the parties participating in the race is *temporary* in nature and is lost once another party overtakes the lead. Two types of models are presented in Golany et al. (2012): optimization models that derive optimal resource allocation schemes, and game-theoretic models that derive Nash-equilibrium solutions. Specifically, resources invested by  $B$  in developing CMs determine, probabilistically, the time when these CMs are ready and operational, and thus also determine the expected damage caused to  $B$  by  $R$ . Assuming a predetermined development policy of CMs—in parallel or sequentially—corresponding convex programming problems are formulated and solution methods are discussed.

The stochastic models in Golany et al. (2012) enable  $B$  to determine optimal investment schemes while capturing uncertain durations of R&D activities and limited intelligence about  $R$ ’s capabilities. The approach taken in this paper is quite different because we focus on developing *deterministic* models to address  $B$ ’s resource allocation problems. The deterministic approach is justified in settings where (1) the CM development efforts do not involve a significant research element and are mainly composed of a sequence of engineering stages whose durations can be forecasted with reasonable accuracy and (2) when there are reliable intelligence reports regarding  $R$ ’s capabilities, intentions, and possible hostile actions.

The main contribution of this paper is in extending the operational situation described in Golany et al. (2012) in three ways: (a) assuming arbitrary CM development policies (not necessarily parallel or sequential); (b) introducing temporal budget constraints, which are quite realistic in defense contracting; (c) allowing for a wide variety of “inconsistent” CMs in the sense that a certain CM may be more effective against weapon I than weapon II, whereas the reverse is true for another CM. Also, unlike the continuous investment levels considered in Golany et al. (2012), the formulation presented herein restricts the investment levels to a finite number of discrete values. Similar discretization was implemented by Burnett et al. (1993) to analyze investment levels among alternative projects related to the natural gas industry in the United States.

We model the decision problem of  $B$  as a variant of a resource-constrained shortest-path (RCSP) problem, where the constraints capture global or temporal budgetary constraints. The RCSP problem is known to be NP-complete, in the ordinary sense—see Garey and Johnson (1979), and we show that our variant is also NP-complete. RCSP problems have been addressed by many authors including Beasley and Christofides (1989), Dumitrescu and Boland (2001), Hassin (1992), and Mehlhorn and

Ziegelmann (2000). In particular, RCSP problems of limited size can be solved through special-purpose algorithms such as those developed in Carlyle and Wood (2003) and Gellermann et al. (2005), or through efficient general-purpose algorithms available in commercial optimization software packages.

To demonstrate the potential usefulness of our RCSP models and analyze their robustness to small data perturbations, we conducted an extensive computational study in which we employed the solver in the MOSEK optimization package. This solver was proven to be quite efficient for realistically sized problem instances of our RCSP models.

The rest of the paper is organized as follows. In §2 we introduce notation and state the decision problem formally. In §§3 and 4 we formulate several variants of the problem, addressing single and multiple weapons and CMs, as constrained network optimization models. Section 5 demonstrates the usefulness of the models by presenting the results of extensive numerical experiments. Finally, §6 suggests some directions for future research.

## 2. Problem Formulation

The weapons that  $R$  develops are indexed by  $w \in \mathcal{W} = \{1, \dots, |\mathcal{W}|\}$ . For each  $w$ , let  $s_w$  denote the time when weapon  $w$  becomes operational; the  $s_w$ s are obtained or estimated by  $B$ ’s intelligence agencies. In particular,  $s_w = 0$  means that weapon  $w$  is already operational at time 0. If  $s_w > 0$ , then weapon  $w$  does not contribute to the damage inflicted on  $B$  until time  $s_w$ . We consider a finite time horizon of length  $T$ . Without loss of generality, we assume that  $T$  is large enough such that all weapons would become operational before time  $T$  and that the weapons are indexed in increasing order of the  $s_w$ s; that is,  $0 \leq s_1 \leq \dots \leq s_{|\mathcal{W}|} \leq T$ . Absent any CM, the damage rate per unit-time inflicted by weapon  $w$  on  $B$  is  $d_0^w \geq 0$ . Absent any weapon, the damage rate per unit-time is 0, independent of the available CMs.

To mitigate the effect of  $R$ ’s weapons,  $B$  develops CMs. These CMs are indexed by  $m \in \mathcal{M} = \{1, \dots, |\mathcal{M}|\}$ , and we use the notation  $\text{CM}_m$  to refer to the  $m$ th CM. For each  $w$  and  $m$ , let  $d_m^w \geq 0$  be the damage rate caused by weapon  $w$  when only  $\text{CM}_m$  is operational. When a set of CMs is available, their effect is not cumulative—the damage rate of weapon  $w$  is determined by the most effective CM that is available at that time. Although in some cases there may be cumulative effects of CMs, e.g., when one CM is a detection device and another CM is a neutralization device, we focus in this paper on a single family of CMs (e.g., interception systems or bomb neutralization systems) that evolves and improves over time and whose members differ in their capabilities. Therefore, when a set  $\emptyset \neq M \subseteq \mathcal{M}$  of CMs is available, the damage rate by weapon  $w$ , if operational, is  $d_M^w \equiv \min_{m \in M} d_m^w$ ; when no CM is available, the damage rate is  $d_\emptyset^w \equiv d_0^w$ . We will find it useful to apply the notation  $d_M \equiv (d_M^1, \dots, d_M^{|\mathcal{W}|}) \in \mathbb{R}^{|\mathcal{W}|}$  for  $M \subseteq \mathcal{M}$ ,  $d_m \equiv d_{\{m\}}$  for  $m \in \mathcal{M}$  and  $d_\emptyset \equiv d_\emptyset$ .

The damage rate caused by a group of weapons is represented by a monotonically increasing function  $D: \mathbb{R}^{|W|} \rightarrow \mathbb{R}$ , which converts damage rates of individual weapons into a total damage rate. By *monotonically increasing* function  $D(x_1, \dots, x_{|W|})$ , we mean that if  $x_w \geq y_w$  for all  $w = 1, \dots, |W|$ , then  $D(x_1, \dots, x_{|W|}) \geq D(y_1, \dots, y_{|W|})$ ; and if  $x_w > y_w$  for all  $w = 1, \dots, |W|$ , then  $D(x_1, \dots, x_{|W|}) > D(y_1, \dots, y_{|W|})$ . For example,  $D(\cdot)$  can be the sum of the individual damage rates or their maximum. When a weapon  $w \in W$  is not operational, its individual contribution to the total damage is set to 0 when evaluating the function  $D(\cdot)$ . Given a set  $W \subseteq \mathcal{W}$ , let  $I^W \in \{0, 1\}^{|W|}$  be the indicator vector of  $W$ , i.e.,  $(I^W)_w = 1$  if  $w \in W$  and  $(I^W)_w = 0$  otherwise. The damage rate caused by the set  $W$  of operational weapons when the set of available CMs is  $M$  is then expressed by

$$d_M^W = D(I^W \circ d_M),$$

where “ $\circ$ ” stands for the Hadamard product. That is, for vectors  $x, y \in \mathbb{R}^{|W|}$ , the Hadamard product  $x \circ y \in \mathbb{R}^{|W|}$  has  $(x \circ y)_w = x_w y_w$  for each  $w \in W$ . In particular, because each  $d_m^w$  represents the damage rate of a certain weapon  $w$  in the presence of CM $_m$ , we have that  $D[I^{(w)} \circ d_m] = d_m^w$ . Also,  $D(0) = 0$ .

It is assumed that each CM can be developed at any one of several levels of intensity that are indexed by  $k \in \mathcal{K} = \{1, \dots, |\mathcal{K}|\}$ .<sup>1</sup> A higher intensity level of development has two effects: first, the development time of the CM is shorter, and therefore it becomes operational sooner, and second, the associated cost is higher. We assume that the intensity of developing a CM does not affect its effectiveness. Intensity 0 indicates no development. For  $m \in \mathcal{M}$  and  $k \in \mathcal{K}$ , let  $t_m^k \geq 0$  denote the time it takes to complete CM $_m$  when developed at intensity level  $k$  and let  $c_m^k \geq 0$  be the corresponding cost. As higher intensity levels are associated with shorter development times and higher costs, we index the intensities so that

$$\begin{aligned} t_m^{|\mathcal{K}|} &\leq t_m^{|\mathcal{K}|-1} \leq \dots \leq t_m^1 \quad \text{and} \\ c_m^{|\mathcal{K}|} &\geq c_m^{|\mathcal{K}|-1} \geq \dots \geq c_m^1 \quad \text{for each } m \in \mathcal{M}. \end{aligned} \quad (1)$$

To avoid degenerate situations, we assume throughout that there are no ties among the  $d_m^w$ s,  $t_m^k$ s and  $c_m^k$ s.

The problem that  $B$  faces is to decide which CMs to develop, at what times to start development, and at what intensity levels. The goal is to minimize the cumulative damage over the time horizon subject to budgetary constraints. The simplest budgetary constraint is a global one where an upper bound, say  $C$ , is prescribed on the total funds that can be spent in developing the CMs. In this case, the entire budget is available at  $t = 0$ . Because there is no reason to defer the development of any CM, the starting time for all the CMs occurs at time  $t = 0$ . However, budgetary constraints may also be temporal where there is an upper bound on expenditure in a certain time period. In such cases, starting times of the development of the CMs become decision variables.

### 3. Consistent CMs

In this section we analyze variants of the resource allocation problem described in §2 under a *consistency* assumption, formally introduced in §3.2. The problems are formulated as constrained network optimization models, which facilitates their efficient solution. In §3.1 we consider the case of a single weapon (which is trivially consistent) under a global budgetary constraint. In §3.2 we extend the analysis to the case of multiple consistent weapons and in §3.3 we address temporal budgetary constraints.

#### 3.1. Single Weapon

Here we assume that there is a single weapon and no temporal budget constraints; in this case, the convention that  $d_m^1 = D[I^{(1)} \circ d_m]$  for  $m \in \mathcal{M}$  assures that  $D(\cdot)$  is the identity, thus,  $d_M^1 = D(d_M) = d_M$  for every  $M \subseteq \mathcal{M}$ . Henceforth, in this subsection we suppress the index  $w = 1$ . Also, we rank the CMs by their effectiveness (against the single weapon), that is,  $d_0 > d_1 > \dots > d_{|\mathcal{M}|} \geq 0$ . Because there are no temporal budget constraints, it can be assumed that the development of all CMs starts at 0. We first assume that the single weapon is already operational at time 0.

The following definitions and notation are used throughout the paper. A *CM-development policy*, henceforth called simply a *policy*, is a set of CMs along with corresponding development intensities. Thus, a policy  $\pi$  is represented by a set of pairs  $(m, k)$  where the values of  $m$  are distinct. We order the pairs in  $\pi$  by their first coordinate such that the sequence  $(m_1, k_1), (m_2, k_2), \dots, (m_p, k_p)$  satisfies  $m_1 < m_2 < \dots < m_p$ . Clearly, there is no point in developing a CM that becomes available *after* a more effective CM is already operational because such a CM cannot reduce the damage caused by the weapon. Consequently, one can restrict attention only to policies  $\pi$  that have the following property: if  $(m, k)$  and  $(m', k')$  with  $m < m'$  are in  $\pi$ , then  $t_m^k < t_{m'}^{k'}$ . We refer to policies that satisfy this condition as *effective policies*.

A *path* in a graph is an ordered set of vertices where each consecutive pair is an edge. A path that starts at vertex  $a$  and ends at vertex  $b$  is referred to as an  $(a - b)$ -path. For a path  $\sigma$ ,  $V(\sigma)$  is the set of vertices in  $\sigma$ , excluding the end vertices, and  $E(\sigma)$  is the set of the corresponding edges.

Effective policies can be represented by paths in the acyclic (directed) graph  $G' = (V', E')$ , where

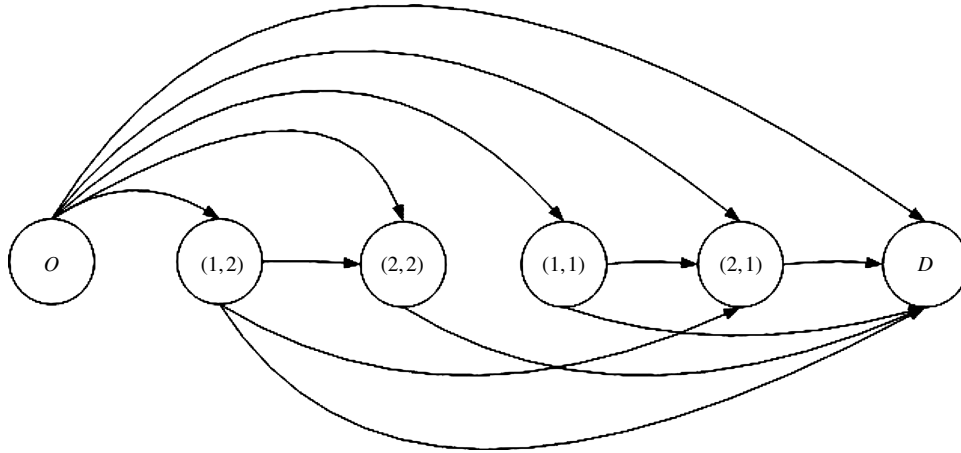
$$V' \equiv \{(m, k) : m \in \mathcal{M} \text{ and } k \in \mathcal{K}\} \quad (2)$$

and

$$E' \equiv \{((m, k), (m', k')) : m < m' \text{ and } t_m^k < t_{m'}^{k'}\}. \quad (3)$$

Next, we augment the graph  $G'$  with a single origin and a single destination, denoted  $O$  and  $D$ , respectively, and with edges  $(O, D)$ ,  $(O, (m, k))$ , and  $((m, k), D)$  for every  $(m, k) \in V'$ . The resulting augmented graph is referred to as the *Feasible CM Development Schedule* (FCDS) graph, and its vertex and edge sets are denoted  $V$  and  $E$ , respectively.

**Figure 1.** An FCDS graph for a single weapon.



It is convenient to represent the vertices of an FCDS graph on the interval  $[0, T]$  with vertices of  $V'$  represented by their corresponding completion times  $t_m^k$ ,  $O$  corresponding to 0 and  $D$  corresponding to  $T$ . In this representation, all edges have orientation from left to right, assuring that the FCDS graph is acyclic and therefore its paths are simple.

Figure 1 presents a situation where developing  $CM_2$  at intensity 2 takes less time than developing  $CM_1$  at intensity 1, and therefore  $((2, 2), (1, 1))$  is not an edge in the corresponding FCDS graph. However, if the same intensity level is applied to  $CM_1$  and  $CM_2$ , then  $CM_1$  is completed before  $CM_2$ —hence,  $((1, 1), (2, 1))$  and  $((1, 2), (2, 2))$  are edges in the FCDS graph.

The discussion above implies that we have a one-to-one correspondence between the set of effective policies and the set of  $(O-D)$ -paths in the FCDS graph. For example, in Figure 1, the effective policy  $\{(1, 2), (2, 1)\}$  is represented by the path  $((O, (1, 2)), ((1, 2), (2, 1)), ((2, 1), D))$ , but there is no path for the noneffective policy  $\{(1, 1), (2, 2)\}$ . We define two transformations denoted  $policy(\cdot)$  and  $path(\cdot)$ , which convert paths to effective policies and vice versa. These transformations are inverses of each other, i.e., if  $\sigma$  is a path in an FCDS and  $\pi$  is a policy, then  $policy(\sigma) = \pi$  if and only if  $path(\pi) = \sigma$ ; in particular,  $policy[path(\pi)] = \pi$  and  $path[policy(\sigma)] = \sigma$ . Although the definitions of  $policy(\cdot)$  and  $path(\cdot)$  may seem trivial, they will be particularly useful in §4.

The FCDS graph has  $|\mathcal{M}||\mathcal{K}| + 2$  vertices and at most  $|\mathcal{K}|^2 \binom{|\mathcal{M}|}{2} + 2|\mathcal{M}||\mathcal{K}| + 1$  edges. The bound on the number of edges is attained when the (time) intervals  $\{[t_m^{\mathcal{K}}], [t_m^1]: m \in \mathcal{M}\}$  are pairwise disjoint, that is, when the development times of the various CMs are highly variable. The actual number of edges of the FCDS graph depends on the  $t_m^k$  values and, in general, the graph may be sparse because of overlaps of the aforementioned time intervals. That is, the completion time of some highly effective CMs developed at high intensity levels may be shorter than that of less-effective CMs that are developed at a low intensity level.

A nonempty effective policy  $\pi$ , represented by a path  $(O, (m_1, k_1), (m_2, k_2), \dots, (m_{p-1}, k_{p-1}), (m_p, k_p), D)$  of the FCDS graph, defines a partition of  $[0, T]$  into the time intervals  $[0, t_{m_1}^{k_1}], [t_{m_1}^{k_1}, t_{m_2}^{k_2}], \dots, [t_{m_{p-1}}^{k_{p-1}}, t_{m_p}^{k_p}], [t_{m_p}^{k_p}, T]$ . During the time interval  $[t_{m_j}^{k_j}, t_{m_{j+1}}^{k_{j+1}}]$  the available protection against the weapon is that of  $CM_{m_j}$ , that is, the damage rate inflicted by the weapon is  $d_{m_j}$ . The total damage during that period is  $d_{m_j}(t_{m_{j+1}}^{k_{j+1}} - t_{m_j}^{k_j})$ . Similarly, the total damage inflicted during the interval  $[0, t_{m_1}^{k_1}]$  is  $d_0 t_{m_1}^{k_1}$ , and the total damage inflicted during the interval  $[t_{m_p}^{k_p}, T]$  is  $d_{m_p}(T - t_{m_p}^{k_p})$ . Thus, by assigning to each edge  $e$  of the FCDS graph a *damage value*  $d_e$  given by

$$d_e \equiv \begin{cases} d_m(t_{m'}^{k'} - t_m^k) & \text{if } e = ((m, k), (m', k')) \\ d_0(t_{m'}^{k'} - 0) & \text{if } e = (O, (m', k')) \\ d_m(T - t_m^k) & \text{if } e = ((m, k), D) \\ d_0(T - 0) & \text{if } e = (O, D), \end{cases} \quad (4)$$

the total damage inflicted in the time interval  $[0, T]$  when an effective policy  $\pi$  is implemented is expressed by

$$d(\pi) \equiv \sum_{e \in E[path(\pi)]} d_e. \quad (5)$$

The next lemma records an *opposite triangular inequality* that the  $d_e$ s (the damage values) satisfy, which we will use later in §4.

**LEMMA 1.** Suppose  $(v_1, v_2)$  and  $(v_2, v_3) \in E$ . Then,  $(v_1, v_3) \in E$  and

$$d(v_1, v_3) > d(v_1, v_2) + d(v_2, v_3). \quad (6)$$

**PROOF.** We prove only for  $v_1, v_2, v_3 \in V^1$ , e.g.,  $v_i = (m_i, k_i)$  for  $i=1,2,3$ . Trivially, the assumptions imply  $m_1 < m_2 < m_3$  and  $t_{m_1}^{k_1} < t_{m_2}^{k_2} < t_{m_3}^{k_3}$ . To verify (6), note that  $d_{((m_1, k_1), (m_3, k_3))} = d_{m_1}(t_{m_3}^{k_3} - t_{m_1}^{k_1}) = d_{m_1}(t_{m_3}^{k_3} - t_{m_2}^{k_2}) + d_{m_1}(t_{m_2}^{k_2} - t_{m_1}^{k_1}) > d_{m_2}(t_{m_3}^{k_3} - t_{m_2}^{k_2}) + d_{m_1}(t_{m_2}^{k_2} - t_{m_1}^{k_1}) = d_{((m_1, k_1), (m_2, k_2))} + d_{((m_2, k_2), (m_3, k_3))}$  (the last inequality follows from  $d_{m_1} > d_{m_2}$ ).  $\square$

The cost  $c_e$  of an edge in the FCDS graph represents the resources needed for developing the CM corresponding to the end-vertex of that edge, that is,

$$c_e \equiv \begin{cases} c_m^k & \text{if the end-vertex of } e \text{ is } (m, k) \\ 0 & \text{if the end-vertex of } e \text{ is } D. \end{cases} \quad (7)$$

The cost of implementing an effective policy  $\pi$  is then expressed by

$$c(\pi) \equiv \sum_{e \in E[\text{path}(\pi)]} c_e. \quad (8)$$

Due to (7), (8), and the one-to-one correspondence of effective policies and paths in the FCDS graph, the problem of selecting the effective policy that minimizes the damage subject to budget constraint  $C$  reduces to the problem of finding an  $(O-D)$ -path in the FCDS graph that minimizes the  $d$ -length (5) subject to the  $c$ -length (8) being bounded by  $C$ , that is,

$$\begin{aligned} \min_{\sigma} \quad & \sum_{e \in E(\sigma)} d_e \\ \text{s.t.} \quad & \sum_{e \in E(\sigma)} c_e \leq C \\ & \sigma \text{ is an } (O-D)\text{-path in the FCDS graph.} \end{aligned} \quad (9)$$

Formally,

**PROPOSITION 1.** *In the case of a single weapon, the vertices  $(m, k)$  on a path that is optimal for (9) determine an (effective) optimal policy that minimizes the damage that  $R$  causes  $B$  subject to the budget constraint.*

Unfortunately, the next result shows that (9) is theoretically hard.

**LEMMA 2.** *The constrained shortest-path problem on an FCDS graph is NP complete.*

**PROOF.** See the appendix.  $\square$

We next relax the assumption that the (single) weapon is available at time 0 and assume that it becomes operational at time  $0 < s < T$ . In this case, the total damage inflicted during a time interval  $[a, b]$  in which only  $\text{CM}_m$  is available is given by  $d_m[\max\{b, s\} - \max\{a, s\}]$ , and the same formula applies with  $m = 0$  if no CM is available. It follows that the total damage inflicted by the weapon when an effective policy  $\pi$  is used is the length of the corresponding path where edge lengths are given by the modification of (4) obtained by replacing:

$$\begin{aligned} t_m^k &\rightarrow \max\{t_m^k, s\} \quad \text{for each } (m, k) \in V \\ 0 &\rightarrow s. \end{aligned} \quad (10)$$

No change is needed in the representation of the cost associated with developing effective policies. It follows that the problem still reduces to the constrained shortest-path problem of (9), and Proposition 1 extends to the case where  $s \neq 0$ .

### 3.2. Multiple Weapons and Consistent CMs

Consider the case where there are multiple weapons threatening  $B$ . We say that the CMs are *consistent* if the rankings of their effectiveness against all the weapons coincide. Formally, this means that the CMs can be indexed so that

$$d_0^w > d_1^w > \dots > d_{|\mathcal{M}|}^w \quad \text{for each } w \in \mathcal{W}. \quad (11)$$

In particular, for any subset  $M \subseteq \mathcal{M}$ ,  $\arg \min_{m \in M} d_m^w$  is invariant of  $w$ , implying that for every  $W \subseteq \mathcal{W}$ ,

$$d_M^W = D(I^W \circ d_M) = \min_{m \in M} D(I^W \circ d_m). \quad (12)$$

The consistency property is applicable when the weapons are similar (e.g., various types of roadside IEDs) and the differences among the CMs are only manifested in the extent of their damage reduction. Henceforth, in this section, we assume that the CMs are consistent. Also, no temporal budget constraints are imposed and the development of all CMs starts at 0.

Recall that a policy is a collection of pairs  $(m, k)$  with distinct values of  $m$ . The definition of effective policies given in §3.1 relies on the ranking of the effectiveness of the CMs. Because the rankings with respect to all weapons are the same, it follows that if  $\text{CM}_m$  is more effective than  $\text{CM}_{m'}$  with respect to one weapon, then this effectiveness dominance applies to all weapons. Consequently, if a policy is effective with respect to one weapon, then it is effective with respect to all weapons.

Again, we start off with the assumption that all the weapons are operational at time 0. The damage inflicted by the weapons when an effective policy  $\pi$  is implemented is a monotone-increasing function of the damages inflicted by the individual weapons. Specifically, and similarly to the analysis in §3.1 (see (4)), the total damage inflicted during the time interval  $[0, T]$  when an effective policy  $\pi$  is implemented is expressed by (5), where

$$d_e \equiv \begin{cases} D(I^{\mathcal{W}} \circ d_m)(t_m^{k'} - t_m^k) & \text{if } e = ((m, k), (m', k')) \\ D(I^{\mathcal{W}} \circ d_0)(t_m^{k'} - 0) & \text{if } e = (O, (m', k')) \\ D(I^{\mathcal{W}} \circ d_m)(T - t_m^k) & \text{if } e = ((m, k), D) \\ D(I^{\mathcal{W}} \circ d_0)T & \text{if } e = (O, D). \end{cases} \quad (13)$$

The expression for the cost  $c(\pi)$  of implementing an effective policy  $\pi$  remains unchanged and is expressed by (7)–(8). As in §3.1, it follows that the problem of selecting the best effective policy reduces to the constrained shortest-path problem of (9). Formally,

**PROPOSITION 2.** *In the case of multiple weapons and consistent CMs, the vertices  $(m, k)$  on a path that is optimal for (9) (with the  $d_e$  values given by (13)) determine an optimal effective policy that minimizes the damage caused by  $R$  to  $B$  subject to the total budget constraint.*

Next, assume that some weapons are not operational at time 0 and weapon  $w$  becomes available at time  $s_w \geq 0$ . Recall (from §2) that the weapons are indexed in increasing order of the  $s_w$ s, i.e.,  $0 \leq s_1 \leq \dots \leq s_{|\mathcal{W}|} \leq T$ ; thus, potential sets of operational weapons that  $B$  may encounter are  $\{1, \dots, w\}$ , where  $w \in \mathcal{W}$ . Using (12), we next observe that  $B$  can continue to restrict attention to effective policies. We adopt the convention that  $[a, b] = \emptyset$  if  $b < a$  and  $r_+ = \max\{r, 0\}$  for a real number  $r$ . Now, if an effective policy  $\pi$  is implemented and  $((m, k), (m', k')) \in E[\text{path}(\pi)]$ , then for  $m \in \mathcal{M}$  and  $w \in \mathcal{W}$ , damage rate  $d_m^{(1, \dots, w)}$  is inflicted during the time interval  $[t_m^k, t_{m'}^{k'}] \cap [s_w, s_{w+1}] = [\max\{t_m^k, s_w\}, \min\{t_{m'}^{k'}, s_{w+1}\}]$ , whose length is  $[\min\{t_{m'}^{k'}, s_{w+1}\} - \max\{t_m^k, s_w\}]_+$ ; when the edge emanates from  $O$ ,  $t_m^k$  is replaced by 0, and when it terminates at  $D$ ,  $t_{m'}^{k'}$  is replaced by  $T$ . Therefore, for  $e \in E$  and  $w \in \mathcal{W}$ , let

$$\tau_{e|w} \equiv \begin{cases} [\min\{t_{m'}^{k'}, s_{w+1}\} - \max\{t_m^k, s_w\}]_+ & \text{if } e = ((m, k), (m', k')) \\ [\min\{t_{m'}^{k'}, s_{w+1}\} - s_w]_+ & \text{if } e = (O, (m', k')) \\ [\min\{T, s_{w+1}\} - \max\{t_m^k, s_w\}]_+ & \text{if } e = ((m, k), D) \\ [\min\{T, s_{w+1}\} - s_w]_+ & \text{if } e = (O, D) \end{cases} \quad (14)$$

and

$$d_e \equiv \begin{cases} \sum_{w \in \mathcal{W}} D[I^{(1, \dots, w)} \circ d_m] \tau_{e|w} & \text{if } e \text{ emanates from } (m, k) \\ \sum_{w \in \mathcal{W}} D[I^{(1, \dots, w)} \circ d_0] \tau_{e|w} & \text{if } e \text{ emanates from } O. \end{cases} \quad (15)$$

The total damage inflicted when an effective policy  $\pi$  is used is then the length of the corresponding path where edge lengths are given by (15). No changes are needed in the representation of the cost associated with the effective policy  $\pi$ . Therefore, the problem is still reduced to the constrained shortest-path problem of (9) and Proposition 2 extends to the case where the  $s_w$ s are not necessarily 0.

### 3.3. Multiple Weapons and Consistent CMs with Temporal Budget Constraints

In this subsection we consider the situation examined in §3.2 with additional constraints that restrict periodic expenditures. We specify  $H$  time intervals (subsets of  $[0, T]$ ) as:  $I_1 = [\underline{T}_1, \bar{T}_1]$ ,  $I_2 = [\underline{T}_2, \bar{T}_2]$ ,  $\dots$ ,  $I_H = [\underline{T}_H, \bar{T}_H]$ , where  $\underline{T}_h < \bar{T}_h$  and assume that there is a bound  $C^h$  on the expenditure during time interval  $I_h$ , for each  $h = 1, 2, \dots, H$ , in addition to the global budget constraint. The time intervals are not necessarily disjoint. There are two special cases of particular interest. In the first case, the intervals  $I_h$  are disjoint and thus partition  $[0, T]$ . In this case the periodic budget constraints represent strict cash

flow constraints (e.g., typical to the U.S. government budgeting rules) in which excess funds in one period cannot be utilized in the next period. Clearly, for the global constraint to be meaningful, the data must satisfy  $\sum_{h=1}^H C^h > C$ . The second case involves relaxed cash flow constraints where budget overflows are allowed to be used in future periods. In this case, the time epochs  $0 = T_1 < T_2 < \dots < T_H < T_{H+1} = T$  are given and  $I_h = [0, T_{h+1}]$  for  $h = 1, \dots, H$ . Here, for the data to be meaningful it must satisfy  $C_1 < C_2 < \dots < C_H < C$  (the constraint corresponding to  $[0, T_{H+1}]$  is left out as it is the global budget constraint).

We assume that costs are incurred continuously and at a uniform rate. Also, the development of CMs is carried out without planned interruptions (an assumption that is quite reasonable because in reality, disrupting a project may incur high set-up cost when development is resumed). In the presence of bounds on periodic expenditures, the time at which the development of each CM starts becomes a decision variable. To address these additional decision variables, we extend the definition of “intensity” and refer to “plans” of CM development projects. Each *plan* consists of a pair  $(k, \tau)$ , where  $k$  is the intensity of developing the CM and  $\tau$  is the start time. For simplicity of exposition, we focus on the case where neither the effectiveness of a CM nor the cost and duration of developing it are affected by  $\tau$ ; relaxation of these assumptions is briefly discussed at the end of this subsection.

As discussed above, when there are no temporal budgetary constraints, there is no reason to defer the development of any CM, and the  $t_m^k$  values represent both the *development duration* and the *completion time*. However, this identity does not hold when the starting times of developing the CMs are not 0. Specifically, if CM <sub>$m$</sub>  is developed using plan  $(k, \tau)$ , then its completion time is  $t_m^{(k, \tau)} \equiv \tau + t_m^k$ . To avoid irrelevant situations, we assume that  $t_m^{(k, \tau)} < T$  for all  $m, k$ , and  $\tau$ .

A *policy*  $\pi$  is now defined as a set of triplets  $(m, k, \tau)$  with distinct values of  $m$ , and effective policies are defined in terms of the  $t_m^{(k, \tau)}$  (the completion times) rather than the  $t_m^k$ s (the duration times). Consider the modification of the FCDS graph, where vertices and edges are defined in terms of the triplets  $(m, k, \tau)$  and the completion times  $t_m^{(k, \tau)}$  instead of the pairs  $(m, k)$  and the duration times  $t_m^k$ . We refer to the resulting graph as the *Timing-Feasible CMs Development Schedule* (T-FCDS) graph. Note that if the development of all CMs can start at any one of  $q$  potential time periods and  $\mathcal{K}$  is the set of potential intensity levels, then the T-FCDS graph has  $q|\mathcal{M}||\mathcal{K}| + 2$  vertices and at most  $(q|\mathcal{K}|)^2 \binom{|\mathcal{M}|}{2} + 2q|\mathcal{M}||\mathcal{K}| + 1$  edges. However, unlike the FCDS graph, the T-FCDS graph may be quite dense in real-world applications. Because an effective policy is such that for any two nodes  $(m, k, \tau)$  and  $(m', k', \tau')$ ,  $m < m'$ , in the T-FCDS graph we have that  $t_m^{(k, \tau)} < t_{m'}^{(k', \tau')}$ , it follows that, as before, there is a one-to-one correspondence between the set of effective policies and the set of  $(O, D)$ -paths in the T-FCDS graph.

Suppose that the development plan of  $CM_m$  is  $(k, \tau)$ . During the development period  $[\tau, t_m^{(k, \tau)}]$  a cost  $\hat{c}_m^k \equiv c_m^k/t_m^k$  per unit time is incurred. The total expenditure on  $CM_m$  during the time interval  $I_h = [\underline{T}_h, \bar{T}_h]$  is then the length of the (possibly empty) interval  $I_h \cap [\tau, t_m^{(k, \tau)}]$  times the per unit-time cost  $\hat{c}_m^k$ . We note that  $I_h \cap [\tau, t_m^{(k, \tau)}] = [\max\{\underline{T}_h, \tau\}, \min\{\bar{T}_h, t_m^{(k, \tau)}\}]$ . Consequently, the expenditure during the time interval  $I_h$  on  $CM_m$  is  $\hat{c}_m^k [\min\{\bar{T}_h, t_m^{(k, \tau)}\} - \max\{\underline{T}_h, \tau\}]_+$ . For each edge  $e$  of the T-FCDS graph and  $h = 0, 1, \dots, H$ , let

$$c_e^h \equiv \begin{cases} \hat{c}_m^k [\min\{\bar{T}_h, t_m^{(k, \tau)}\} - \max\{\underline{T}_h, \tau\}]_+ & \text{if } e \text{ terminates at } (m, k, \tau) \\ 0 & \text{if } e \text{ terminates at } D. \end{cases} \quad (16)$$

The total cost associated with effective policy  $\pi$  during the time-interval  $I_h$  is then expressed by  $\sum_{e \in E[\text{path}(\pi)]} c_e^h$  and this sum is subject to the corresponding temporal budget constraint  $C^h$ .

We next consider the total cost associated with implementing effective policy  $\pi$ . Similarly to the case of a single weapon presented in §3.1, here the total cost is expressed by (8), with the  $c_e$ s given by (7), except that  $(m, k)$  is replaced by  $(m, k, \tau)$ .

Now, consider the total damage associated with an effective policy  $\pi$ . First, assume that all weapons are operational at time 0. For each edge  $e$  of the T-FCDS graph, let  $d_e$  be given by the variant of (13) in which  $k$  and  $k'$  are replaced by  $(k, \tau)$  and  $(k', \tau')$ , respectively. The total damage associated with policy  $\pi$  is then expressed by (5). When weapon  $w$  becomes operational at time  $s_w > 0$ , let  $d_e$  be given by (14) and (15), with  $(k, \tau)$  replacing  $k$ .

The above discussion demonstrates that the problem of selecting a policy that minimizes the total damage subject to total and temporal budget constraints reduces to the problem of finding an  $(O-D)$ -path in the T-FCDS graph that minimizes the  $d$ -length subject to corresponding constraints on the  $c$ -length and the  $c^h$ -lengths, that is,

$$\begin{aligned} \min_{\sigma} \quad & \sum_{e \in E[\sigma]} d_e \\ \text{s.t.} \quad & \sum_{e \in E[\sigma]} c_e \leq C \\ & \sum_{e \in E[\sigma]} c_e^h \leq C^h \quad \text{for } h = 1, \dots, H \\ & \sigma \text{ is an } (O-D)\text{-path in the T-FCDS graph.} \end{aligned} \quad (17)$$

Formally,

**PROPOSITION 3.** *In the case of multiple weapons and consistent CMs with temporal budget constraints, the vertices  $(m, k, \tau)$  on a path that is optimal for (17) determine an optimal effective policy that minimizes the damage caused by  $R$  to  $B$  subject to the total and temporal budget constraints.*

We note that our model can be easily modified to capture situations where costs and development times of CMs depend on the starting times of their development; all that is needed is to replace the parameters  $c_m^k$  and  $t_m^k$  by start-time-dependent counterparts  $c_m^k(\tau)$  and  $t_m^k(\tau)$ , respectively (in which case  $t_m^{k, \tau} = \tau + t_m^k(\tau)$ ). Also, allowing effectiveness of the CMs to depend on the starting times can be captured by replacing  $m$  with  $(m, \tau)$  in the modification of (13).

## 4. Inconsistent CMs

In this section we relax the consistency assumption, allowing for one CM to be more effective than another with respect to weapon  $w$ , whereas the reverse holds for weapon  $w' \neq w$ . Inconsistency is present, for example, when the weapons of  $R$  are not technologically or operationally similar. We analyze the inconsistent case, only when the function  $D(\cdot)$  is the summation function, in which case  $D(I^W \circ d_M) = \sum_{w \in W} d_M^w$ . We start our analysis under the assumptions that all weapons are operational at time 0 and there are no temporal budgetary constraints.

Without the consistency assumption, the CMs can no longer be ranked uniformly according to their effectiveness against the weapons. Still, each weapon has its own total order regarding the effectiveness of the CMs, which we call  $w$ -domination and denote by  $<_w$ ; thus, we write  $m <_w m'$  if  $d_{m'}^w < d_m^w$ .

### 4.1. Policies and Their Effective Parts

We return to the definition in which a policy  $\pi$  is a set of pairs  $(m, k)$  with distinct values of  $m$ . A policy is  $w$ -effective if it is effective in the sense of the definition in §3.1 for weapon  $w$ . Given a policy  $\pi$  and a weapon  $w \in \mathcal{W}$ , the  $w$ -effective part of  $\pi$ , denoted  $\pi_w$ , is the subset of  $\pi$  obtained by removing all pairs  $(m', k') \in \pi$  for which there exists  $(m, k) \in \pi$  with  $m' <_w m$  and  $t_m^k < t_{m'}^{k'}$ . Because a policy applies at most one intensity level for each CM, the  $\pi_w$ s must satisfy the following “coupling requirement”:

$$[(m, k) \in \pi_w \text{ and } (m, k') \in \pi_w] \Rightarrow [k = k']. \quad (18)$$

The total cost of implementing a policy is expressed by

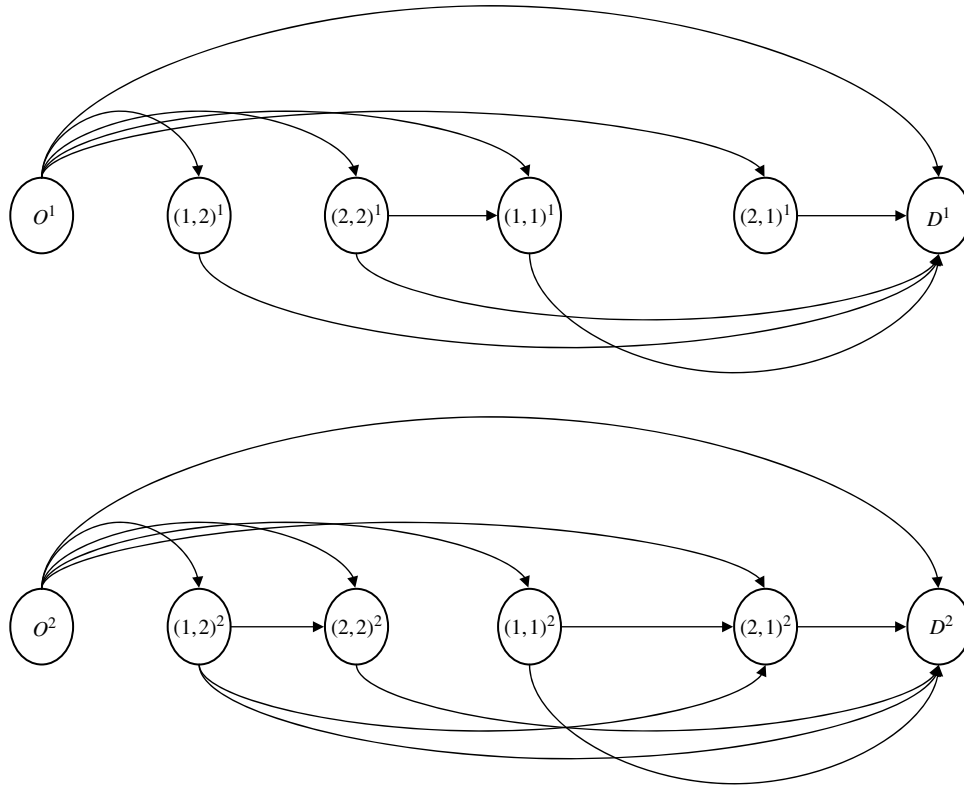
$$c(\pi) \equiv \sum_{(m, k) \in \pi} c_m^k. \quad (19)$$

### 4.2. The Multi-FCDS Graph

Each weapon  $w$  defines an FCDS graph, denoted  $FCDS_w$ , where  $<$  in (3), with respect to the CM indices, is replaced by  $<_w$ . The sets of vertices and edges of the  $FCDS_w$  graph are denoted  $V^w$  and  $E^w$ , respectively. Each  $V^w$  is a replica of  $U \equiv \{O, D\} \cup \{(m, k) : m \in \mathcal{M}, k \in \mathcal{K}\}$ ; in particular, the elements of  $V^w$  are denoted  $O^w$ ,  $D^w$ , and  $(m, k)^w$ . Because the CMs are inconsistent and an edge  $((m, k)^w, (m', k')^w)$  exists in  $E^w$  if and only if  $m <_w m'$  and  $t_m^k < t_{m'}^{k'}$ , it is possible that  $((m, k)^w, (m', k')^w) \in E^w$  but  $((m, k)^{w'}, (m', k')^{w'}) \notin E^{w'}$  for  $w' \neq w$  (this will happen



**Figure 2.** A multi-FCDS graph for two weapons.



if and only if  $t_m^k < t_{m'}^{k'}$ ,  $m \prec_w m'$ , and  $m' \prec_{w'} m$ . As  $((m, k)^w, (m', k')^w) \in E^w$  implies that  $m \prec_w m'$ , the  $\text{FCDS}_w$  graphs are acyclic, and therefore their paths are simple.

In order to refer to the  $\text{FCDS}_w$  graphs jointly, we define the *multi-FCDS* graph  $(V, E)$  whose vertex and edge sets are, respectively,  $V \equiv \bigcup_{w \in \mathcal{W}} V^w$  and  $E \equiv \bigcup_{w \in \mathcal{W}} E^w$ . The  $\text{FCDS}_w$  subgraphs are disjoint components of the multi-FCDS graph, in particular,  $(O^w - D^w)$ -paths of the  $\text{FCDS}_w$  graph are identified with the  $(O^w - D^w)$ -paths of the multi-FCDS graph. Figure 2 presents a multi-FCDS graph for two weapons. As in Figure 1, here too,  $t_1^2 < t_2^2 < t_1^1 < t_2^1$ . However, whereas  $\text{CM}_1$  is more effective than  $\text{CM}_2$  with respect to weapon 1, the reverse is true for weapon 2.

Results in §3.1 show that there is a one-to-one correspondence of  $w$ -effective policies and  $(O^w - D^w)$ -paths of the  $\text{FCDS}_w$  graph, which are the  $(O^w - D^w)$ -paths of the multi-FCDS graph. Following the notation introduced in §3.1, the  $(O^w - D^w)$ -path corresponding to a  $w$ -effective policy  $\pi_w$  is denoted  $\text{path}(\pi_w)$ . Figure 3 demonstrates the  $w$ -effective parts of a policy. Here,  $\pi = \{(1, 1), (2, 2)\}$ ; its 1-effective part is  $\{(1, 1), (2, 2)\} (= \pi)$ , whereas its 2-effective part is only  $\{(2, 2)\}$ .

For each  $e \in E$ , let

$$d_e \equiv \begin{cases} d_m^w(t_{m'}^{k'} - t_m^k) & \text{if } e = ((m, k)^w, (m', k')^w) \\ d_0^w t_{m'}^{k'} & \text{if } e = (O^w, (m', k')^w) \\ d_m^w(T - t_m^k) & \text{if } e = ((m, k)^w, D^w) \\ d_0^w T & \text{if } e = (O^w, D^w). \end{cases} \quad (20)$$

Similar to (5) in §3.1, we then have that the damage that weapon  $w \in \mathcal{W}$  causes when policy  $\pi$  is implemented is expressed by  $d(\pi_w) \equiv \sum_{e \in E[\text{path}(\pi_w)]} d_e$ . The total damage associated with the implementation of policy  $\pi$  (caused by all weapons) is then expressed by

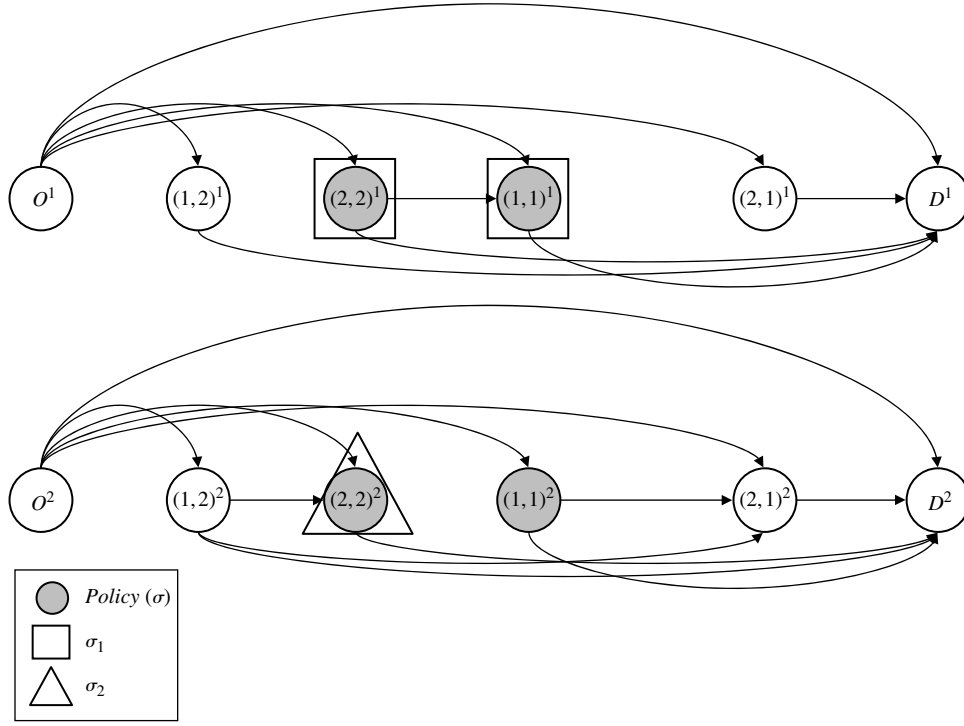
$$d(\pi) \equiv \sum_{w \in \mathcal{W}} d(\pi_w) = \sum_{w \in \mathcal{W}} \sum_{e \in E[\text{path}(\pi_w)]} d_e. \quad (21)$$

### 4.3. Configurations

Unlike the one-to-one correspondence of effective policies and paths in §§3.1–3.3, the relation between policies and paths in the inconsistent case is more complex because the  $\pi_w$ s of a policy  $\pi$  can differ from each other. To overcome this difficulty, we define a *configuration* to be a collection  $\sigma = \{\sigma_w : w \in \mathcal{W}\}$ , where each  $\sigma_w$  is an  $(O^w - D^w)$ -path in the  $\text{FCDS}_w$  subgraph of the multi-FCDS graph. Such a configuration is called *plausible* if it satisfies the “coupling requirement”

$$[(m, k)^w \in V(\sigma_w) \text{ and } (m, k')^{w'} \in V(\sigma_{w'})] \Rightarrow [k = k']. \quad (22)$$

Note that (18) implies that the paths corresponding to the  $w$ -effective parts of a policy  $\pi$  form a plausible configuration. On the other hand, given a plausible configuration  $\sigma = \{\sigma_w : w \in \mathcal{W}\}$ , we define  $\text{policy}(\sigma) \equiv \bigcup_{w \in \mathcal{W}} V(\sigma_w)$ . Finally, we point out that policy  $\pi$  consists of vertices from

**Figure 3.** A representation of the  $w$ -effective parts of a policy.

the multi-FCDS graph that may be replications of  $(m, k)$  pairs, e.g., correspondence between configurations and vectors  $(m, k)^w$  and  $(m, k)^{w'}$ ,  $w = w'$ . The plausibility of  $\sigma$  assures that  $policy(\sigma)$  is indeed a policy. The transformations  $\pi \rightarrow \{path(\pi_w): w \in \mathcal{W}\}$  and  $\sigma \rightarrow policy(\sigma)$  are not inverses of each other.<sup>2</sup>

For a policy  $\pi$  we define

$$\bar{\pi} \equiv policy(\{path(\pi_w): w \in \mathcal{W}\}); \quad (23)$$

note that  $\bar{\pi} \subseteq \pi$  and

$$c(\pi) = \sum_{(m,k) \in \pi} c_m^k \geq \sum_{(m,k) \in \bar{\pi}} c_m^k = c(\bar{\pi}); \quad (24)$$

strict inclusion  $\bar{\pi} \subset \pi$  and strict inequality in (24) are possible when  $\pi$  contains “superfluous”  $(m, k)$ s. Also, for each path  $\sigma_w$  of a configuration  $\sigma$ ,  $V(\sigma_w) \subseteq [policy(\sigma)]_w$  and therefore, by the opposite triangular inequality (Lemma 1),

$$\sum_{e \in E(\sigma_w)} d_e \geq \sum_{e \in E[path(policy(\sigma)_w)]} d_e. \quad (25)$$

Combining (25) with (23) shows that for every configuration  $\sigma = \{\sigma_w: w \in \mathcal{W}\}$ :

$$\begin{aligned} d[policy(\sigma)] &= \sum_{w \in \mathcal{W}} \sum_{e \in E[path(policy(\sigma)_w)]} d_e \\ &\leq \sum_{w \in \mathcal{W}} \sum_{e \in E(\sigma_w)} d_e. \end{aligned} \quad (26)$$

Figure 4 demonstrates a situation with strict inequality in (26). The figure illustrates a plausible configuration  $\sigma$  with  $\sigma_1 = (O^1, (1, 1)^1, D^1)$  and  $\sigma_2 = (O^2, (2, 2)^2, D^2)$  (the edges of these paths are marked in bold). The policy corresponding to  $\sigma$  is  $\pi = \{(1, 1), (2, 2)\}$  and the path corresponding to the 1-effective part of  $\pi$  is  $(O^1, (2, 2)^1, (1, 1)^1, D^1)$ , with edges  $(O^1, (2, 2)^1)$  and  $((2, 2)^1, (1, 1)^1)$  (marked by dotted arrows) replacing edge  $(O^1, (1, 1)^1)$  of  $\sigma_1$ . Due to the opposite triangular inequality,  $\{(1, 1), (2, 2)\}$  provides better protection against weapon 1 than  $\{(1, 1)\}$ —the 1-effective policy corresponding to  $\sigma_1$ .

#### 4.4. Network Optimization with Side Constraints

For an edge  $e$  and vertex  $v$  of the multiple-FCDS, we write  $e \leftarrow v$  if  $e$  emanates from  $v$  and  $e \rightarrow v$  if  $e$  terminates at  $v$ . Standard results of network modeling imply a one-to-one correspondence between configurations and vectors  $x = (x_e)_{e \in E}$  that satisfy:

$$\sum_{e \leftarrow (m,k)^w} x_e = \sum_{e \rightarrow (m,k)^w} x_e \quad \text{for each } m \in \mathcal{M}, k \in \mathcal{K} \quad \text{and } w \in \mathcal{W} \quad (27)$$

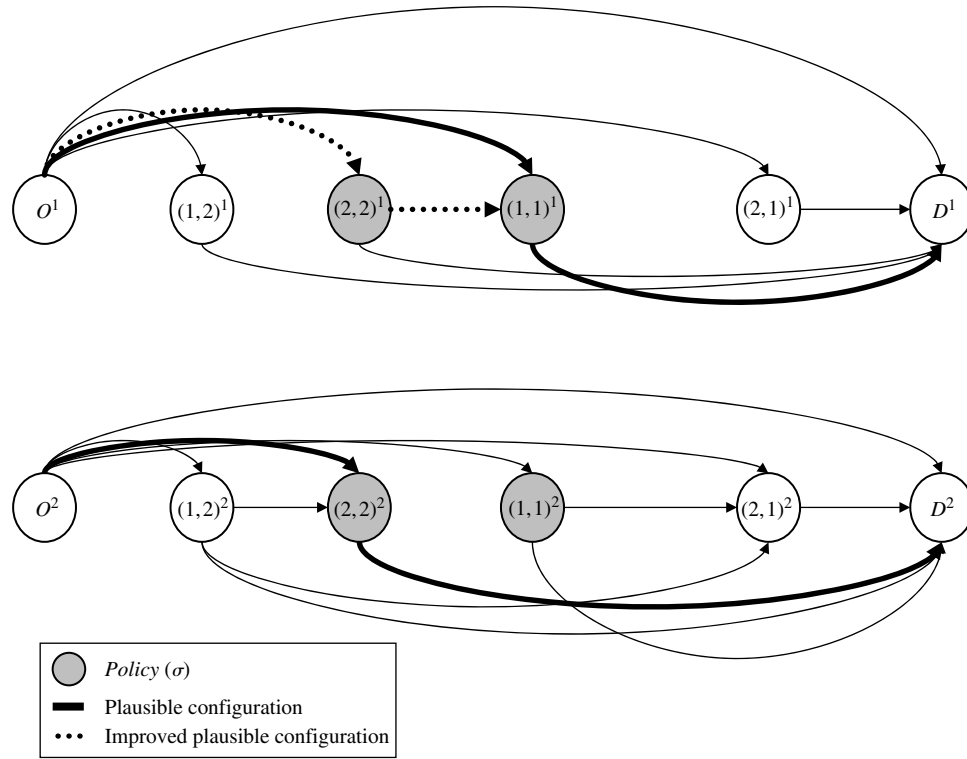
$$\sum_{e \leftarrow O^w} x_e = 1 = \sum_{e \rightarrow D^w} x_e \quad \text{for each } w \in \mathcal{W}$$

$$x_e \in \{0, 1\} \quad \text{for each } e \in E.$$

In particular, if configuration  $\sigma = \{\sigma_w: w \in \mathcal{W}\}$  corresponds to  $x$  satisfying (27), then

$$\sum_{w \in \mathcal{W}} \sum_{e \in E(\sigma_w)} d_e = \sum_{e \in E} d_e x_e. \quad (28)$$

**Figure 4.** Demonstrating opposite triangular inequality.



Given such a solution  $x$ , let  $y(x) = [y(x)_m^k]_{(m,k) \in U}$  be defined by

$$y(x)_m^k = \max_w \sum_{e \rightarrow (m,k)^w} x_e \quad \text{for each } (m,k) \in U;$$

equivalently,  $y(x)$  is the unique solution of

$$y(x)_m^k \geq \sum_{e \rightarrow (m,k)^w} x_e \quad \text{for each } (m,k) \in U \text{ and } w \in \mathcal{W}$$

$$y(x)_m^k \leq \sum_{w \in \mathcal{W}} \sum_{e \rightarrow (m,k)^w} x_e \quad \text{for each } (m,k) \in U$$

$$y(x)_m^k \in \{0, 1\} \quad \text{for each } (m,k) \in U.$$

We note that the variable  $y(x)_m^k$  gets the value 1 if for any weapon  $w$ ,  $\text{CM}_m$  is developed at level  $k$  under  $x$  when considering the  $w$ -subgraph of the multi-FCDS graph: in all other cases  $y(x)_m^k$  gets the value 0.

Consider a configuration  $\sigma$  corresponding to the vector  $x = (x_e)_{e \in E}$  that satisfies (27). Evidently,  $\sigma$  is plausible if and only if  $\sum_{k \in \mathcal{K}} y(x)_m^k \leq 1$  for each  $m \in \mathcal{M}$ ; an equivalent condition is that for some vector  $y = (y_m^k)_{(m,k) \in U}$ ,  $x$  and  $y$  satisfy

$$\begin{aligned} y_m^k &\geq \sum_{e \rightarrow (m,k)^w} x_e && \text{for each } (m,k) \in U \text{ and } w \in \mathcal{W} \\ y_m^k &\leq \sum_{w \in \mathcal{W}} \sum_{e \rightarrow (m,k)^w} x_e && \text{for each } (m,k) \in U \\ \sum_{k \in \mathcal{K}} y_m^k &\leq 1 && \text{for each } m \in \mathcal{M} \\ y_m^k &\in \{0, 1\} && \text{for each } (m,k) \in U. \end{aligned} \quad (29)$$

Thus, we have a one-to-one correspondence between the set of plausible configurations and the set of solutions of (27) and (29). Further, if  $(x, y)$  satisfying (27) and (29) corresponds to a plausible configuration  $\sigma$ , then  $\text{policy}(\sigma) = \{(m, k): y_m^k = 1\}$  and

$$c[\text{policy}(\sigma)] = \sum_{(m,k) \in U} c_m^k y_m^k; \quad (30)$$

the budget constraint on  $\text{policy}(\sigma)$  is then expressed by

$$\sum_{(m,k) \in U} c_m^k y_m^k \leq C. \quad (31)$$

We next show that the problem of selecting an optimal policy reduces to solving the optimization problem:

$$\begin{aligned} \min_{x, y} \quad & \sum_{e \in E} d_e x_e \\ \text{s.t.} \quad & (x, y) \text{ satisfies (27), (29), and (31).} \end{aligned} \quad (32)$$

**PROPOSITION 4.** *In a setting that possibly involves inconsistent weapons, let  $(x^*, y^*)$  be an optimal solution of (32) and  $\pi^* \equiv \{(m, k) \in U: (y^*)_m^k = 1\}$ . Then  $\pi^*$  is a policy that minimizes the damage caused by  $R$  to  $B$  subject to the total budget constraint.*

**PROOF.** From the definition of  $\pi^*$ ,

$$c(\pi^*) = \sum_{(m,k) \in U} c_m^k (y^*)_m^k \leq C, \quad (33)$$

assuring that  $\pi^*$  satisfies the budget constraint. Next, let OPT be the optimal value of (32). With  $\sigma^* = \{\sigma_w^*: w \in \mathcal{W}\}$  as the plausible configuration corresponding to  $(x^*, y^*)$ , we have that  $\pi^* = \text{policy}(\sigma^*)$ ; it then follows from (26) and (28) that

$$\begin{aligned} d(\pi^*) &= d[\text{policy}(\sigma^*)] \\ &\leq \sum_{w \in \mathcal{W}} \sum_{e \in E(\sigma_w^*)} d_e = \sum_{w \in \mathcal{W}} \sum_{e \in E} d_e x_e^* = \text{OPT}. \end{aligned} \quad (34)$$

To see that  $\pi^*$  is optimal, consider an arbitrary policy  $\pi$  that satisfies the total budget constraint. Let  $(x, y)$  be the solution of (27) and (29) that corresponds to the plausible configuration  $\sigma \equiv \{\text{path}(\pi_w): w \in \mathcal{W}\}$ . Then  $\text{policy}(\sigma) = \bar{\pi}$  and (30) and (24) imply that

$$\sum_{(m,k) \in U} c_m^k y_m^k = c(\bar{\pi}) \leq c(\pi) \leq C,$$

assuring that  $(x, y)$  satisfies (31). Therefore,  $(x, y)$  is feasible for (32), and therefore  $\text{OPT} \leq \sum_{e \in E} d_e x_e$ . It now follows from (34), (28), and (21) that

$$d(\pi^*) \leq \text{OPT} \leq \sum_{e \in E} d_e x_e = \sum_{w \in \mathcal{W}} \sum_{e \in E[\text{path}(\pi_w)]} d_e = d(\pi). \quad \square$$

We next consider the case where weapons are not necessarily operational at time 0, and for each  $w \in \mathcal{W}$ , weapon  $w$  becomes operational at time  $0 \leq s_w \leq T$ . The situation is handled by adjusting the definition of the damage coefficients given in (20) (resembling the use of (10) to modify (4) in §3.1). Specifically, for  $m \in \mathcal{M}$ ,  $k \in \mathcal{K}$ , and  $w \in \mathcal{W}$ , let  $t_m^{k,w} \equiv \max\{t_m^k, s_w\}$  and for each  $e \in E$ , let

$$d_e \equiv \begin{cases} d_m^w(t_m^{k',w} - t_m^{k,w}) & \text{if } e = ((m, k)^w, (m', k')^w) \\ d_0^w(t_m^{k',w} - s_w) & \text{if } e = (O^w, (m', k')^w) \\ d_m^w(T - t_m^{k,w}) & \text{if } e = ((m, k)^w, D^w) \\ d_0^w(T - s_w) & \text{if } e = (O^w, D^w). \end{cases}$$

With this adjustment, the total damage inflicted when an effective policy  $\pi$  is used is the sum of the length of the corresponding paths, the decision problem of  $B$  reduces to (32) and Proposition 4 applies. It is also possible to generalize (32) to the case where periodic budget constraints are imposed (see §3.3). Although the model remains essentially the same, its size increases significantly. The numerical experiments reported in the next section apply to this more general case with cash flow budget constraints. That is, the intervals  $I_h$  partition  $[0, T]$  with time epochs  $0 = T_1 < T_2 < \dots < T_H < T_{H+1} = T$ , and each interval  $I_h$  corresponds to the period  $[T_h, T_{h+1}]$ .

## 5. Numerical Experiments

In order to examine the applicability and robustness of our constrained network model, we conducted an extensive computational study. In this study, we implemented a model with inconsistent weapons (see §4), multiperiod temporal budget constraints (see §3.3) where the temporal budget constraints correspond to  $H$  disjoint intervals that partition  $[0, T]$ , and  $D(\cdot)$  as the summation function (see the last paragraph of §4.4). A detailed formulation of the model implemented in our numerical study is given in the appendix.

The case study consisted of 10 base cases and the generation and solution of 100 instances for each of the base cases. Each instance represented a small perturbation of the development completion times of the CMs. The specific model we address is provided in the appendix. All computational tests were carried out on a laptop computer with a Genuine Intel 1 GHz T2500 processor and 1 GB RAM, running the Red Hat Linux 5 operating system. The code was built with C++ version 4.1.1 and linked with glibc 2.5.4 and Mosek 5.0. Each one of the 10 base cases represented 10 types of (inconsistent) weapons ( $|\mathcal{W}| = 10$ ), 10 possible CMs ( $|\mathcal{M}| = 10$ ), 3 levels of intensity ( $|\mathcal{K}| = 3$ ), and 4 time periods ( $H = 4$ ). Consequently, each integer programming formulation has about 38,000 variables and 73,000 constraints and the corresponding multi-FCDS graph has about 120 vertices (one vertex for each possible completion time  $t_m^{(k,\tau)}$ ) and 4,000 edges. Although the problems are quite large, the model proved to be computationally efficient; the average running time of each instance was about 15 minutes (the shortest time was just a few seconds and the longest time was a little over 4 hours).

### 5.1. Parameters Setting

The CMs were divided into three groups according to their completion times: the first group (consisting of CMs indexed by  $m = 1, 2, 3$ ) had short development times, the second group (consisting of CMs indexed by  $m = 4, 5, 6$ ) had medium development times, and the third group (consisting of CMs indexed by  $m = 7, 8, 9, 10$ ) had long development times. The time units were measured in months, and each time period was set to be 12 months. Therefore, the first period is  $[0, 12]$ , the second  $[12, 24]$ , etc.; thus,  $T_2 = 12$ ,  $T_3 = 24$ , and  $T_4 = 36$ . The time horizon,  $T$ , was defined to be  $T_4 + \max_{k,m} \{t_m^k\}$ . For simplicity, in our computational testings we limit  $\tau$ , the optional starting times of developing the CMs to take the values of  $0 = T_1, T_2, T_3$ , or  $T_4$ . The completion times, the  $t_m^k$ s, were sampled uniformly from the data listed in Table 1.

For each CM, the development completion times were ordered according to the three intensities such that  $t_m^1 > t_m^2 > t_m^3$ . The damage rates, the  $d_m^w$ s, were sampled from a trimmed Normal distribution with parameters,  $\mu = 2$ ,  $\sigma = 1.5$  and truncated by 10 and 0 as upper and lower bounds, respectively. Recall that in the case of inconsistent

**Table 1.** Values for the  $t_m^k$ s.

$m$	$k$		
	1	2	3
1, 2, 3	4, 6, 8	10, 12, 14	16, 18, 20
4, 5, 6	12, 15, 18	21, 24, 27	30, 33, 36
7, 8, 9, 10	24, 28, 32, 36	40, 44, 48	52, 56, 60

CMs, the  $d_m^w$ s do not have the same order of the  $m$ s for all  $w \in W$ ; still, for each  $w$ ,  $d_0^w$  is the maximum over all  $d_m^w$ . In addition,  $s_w$ , the time weapon  $w$  is expected to be operational, was sampled from the Uniform distribution on the interval  $[0, 15]$ , and these times were sorted in an ascending order. The development costs  $c_m^k$  were sampled from the Uniform distribution on the interval  $[3, 10]$  and for each  $m$ , the  $c_m^k$ s were ordered to be nondecreasing in  $k$ . The budget temporal constraints for all time periods were set to be identical for all  $h$ , so that  $C^h = \hat{C}$  for all  $h$ .  $\hat{C}$  was calculated as the average costs multiplied by twice the average period's duration  $\delta$ , that is:

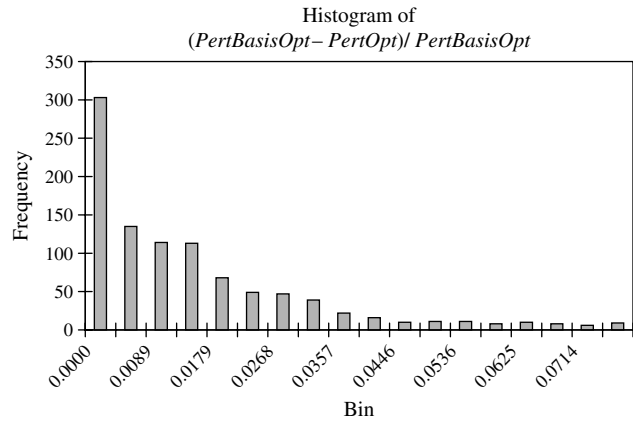
$$\hat{C} = \frac{\sum_{k \in \mathcal{K}} \sum_{m \in \mathcal{M}} (c_m^k / t_m^k)}{|\mathcal{K}| \cdot |\mathcal{M}|} \cdot 2 \cdot \delta,$$

where  $\delta = \sum_{h=1}^H (T_{h+1} - T_h) / H$ . The budget constraint for the entire time period  $T$  was taken as  $C = \hat{C} \cdot H \cdot 0.9$ .

## 5.2. Testing Robustness

As indicated before, an experiment was carried out in order to test the robustness of the solutions to mild perturbations in the data. In particular, we investigated the sensitivity to small deviations in the development completion times. First, the 10 base cases were solved. We next generated 100 instances for each base case, with each instance representing a mild perturbation of the completion times of the base case. The instances were generated by sampling from a trimmed Normal distribution. Specifically, for the completion time of the  $m$ th CM under the  $k$ th intensity we used a Normal distribution  $(\mu, \sigma)$  with  $\mu = t_m^k$  and  $\sigma$  determined in the following way: for  $m = 1, 2, 3$ :  $\sigma = 0.05 \cdot 14$ ; for  $m = 4, 5, 6$ :  $\sigma = 0.05 \cdot 24$ ; and for  $m = 7, 8, 9, 10$ :  $\sigma = 0.05 \cdot 44$ . The range of the trimmed Normal distribution was  $[\mu - 3\sigma, \mu + 3\sigma]$ . The completion times were not reordered to be monotone in the development intensity level. The time intervals and the damage costs, which are functions of the completion times, were recalculated in the same way as in the base case, whereas the  $C_i$ s were not recalculated. Each of the 100 instances of each base case was solved separately.

**Figure 5.** Histogram of the data in Table 2.



## 5.3. Measurements

We tested the robustness of the solution relative to each base case  $j \in \{1, \dots, 10\}$  and each perturbation  $i \in \{1, \dots, 100\}$ . For each  $j$  and  $i$ , let  $PertOpt_{ji}$  be the optimal value (total damage) of the  $i$ th perturbation of the  $j$ th base problem, let  $(X_j^{opt}, Y_j^{opt})$  be an optimal solution of the  $j$ th base case, and let  $PertBasisOpt_{ji}$  be the total damage incurred if the actual development completion times are as in the  $j$ th instance and  $(X_j^{opt}, Y_j^{opt})$  is used. We consider the following measure of robustness:

$$\rho_{ji} = \frac{(PertBasisOpt_{ji} - PertOpt_{ji})}{PertBasisOpt_{ji}}.$$

Note that  $(X_j^{opt}, Y_j^{opt})$  might be an infeasible solution to the  $i$ th perturbation of the  $j$ th base case. Infeasibility might be of two types—violation of the topology or violation of the temporal budget constraints. Topological violation can occur if some of the edges in the solution corresponding to  $(X_j^{opt}, Y_j^{opt})$  do not exist in the graph corresponding to the  $j$ th instance. Such violations were handled by direct calculation of the actual value of the solution of the  $j$ th instance. We did not calculate the  $\rho_{ji}$ s for instances that were determined to be infeasible due to violations of the temporal budget constraints.

## 5.4. Numerical Results

Table 2 and its histogram in Figure 5 illustrate the frequencies of the  $\rho$  values obtained from 792 feasible solutions out of the 1,000 runs (in 208 runs there were mild violations in the budget constraints, as indicated in Tables 3 and 4).

We further examined the effect of the temporal budget violations in the following way. For each infeasible

**Table 2.** Frequency of the  $\rho$  values in the original runs.

Value	0	0.004	0.009	0.013	0.018	0.022	0.027	0.031	0.036	0.040	more
Frequency	456	127	86	39	25	24	16	12	5	2	1

**Table 3.**  $\rho$  values with the original temporal budget constraints.

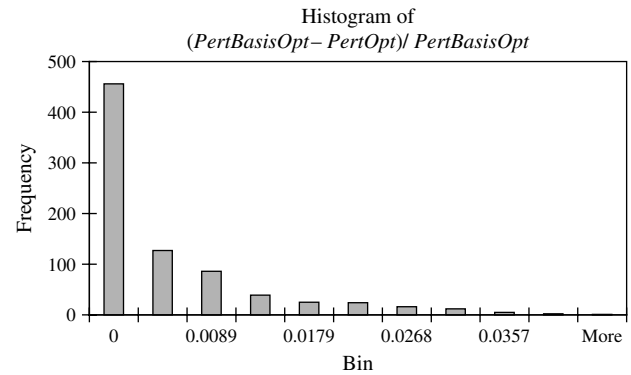
Count	Average	STD	Max	Min
792	0.0040	0.0073	0.0446	0

perturbation, we reran the model with an increase of 5% in each violated temporal budget constraint. The purpose of this exercise was twofold—first, to see whether a relatively small increase in the budget (5%) is sufficient to restore feasibility, and second, to see what effect that increase will have on the  $\rho$  values. Table 4 and its histogram in Figure 6 provide the resultant  $\rho$  values after the 5% increase was implemented where necessary, and Table 5 provides the new summary statistics. Comparing Table 3 to Table 5, we observe that the small increase in budget was sufficient to eliminate almost all the cases of infeasibility (979 out of the 1,000 instances are now feasible). Thus, we conclude that the violations of the temporal budget constraints were of small magnitude. Also, we note that the average  $\rho$  value remains rather small (0.0135). This exercise demonstrates the robustness of our model.

## 6. Concluding Remarks and Extensions

This paper addresses a resource allocation problem faced by a defense agency charged with developing CMs in a dynamic arms race against an adversary that seeks to cause as much damage as possible. Such scenarios are becoming more and more relevant in asymmetric wars between government forces and insurgents. The main contribution in this paper is the formulation of tractable network optimization models that encompass the essential elements of the problem. The models we develop are deterministic, and as such they can be criticized for failing to address the (obvious) uncertainty that exists in the development times of the CMs. However, the extensive numerical analysis that is presented in §5 demonstrates the robustness of the models whose outcomes remain stable when there is some “noise” in the data.

The methods we developed apply to a modification of our model where the data consists of damage rates  $d_m^W$  for  $m \in \mathcal{M}$  and  $W \subseteq \mathcal{W}$  and the damage rate in the presence of a set of weapons  $W$  and a set of CMs  $M$  is  $D_M^W \equiv \min_{m \in \mathcal{M}} d_m^W$  (in the presence of consistency, (12) shows that the model we study is an instance of the above). When all weapons are operational at time 0, this model reduces

**Figure 6.** Histogram of the data in Table 4.

to the one studied in §3.1 by looking at  $\mathcal{W}$  as a single weapon. When the weapons are not necessarily operational at time 0 and weapon  $w$  becomes available at time  $s_w \geq 0$ , with  $0 \leq s_1 \leq \dots \leq s_{|\mathcal{W}|}$ , we can impose a modified consistency assumption that asserts that the ranking of the CMs against the sets of the form  $\{1, \dots, w\}$ ,  $w \in \mathcal{W}$ , is the same. Under this assumption, the analysis of §3.2 applies, and the decision problem can be reduced to (9).

In future research we intend to explore dynamic versions of the models developed here. In particular, we intend to look at multistage models with recourse. That is, in each period,  $B$  will be able to observe new data that were realized since his previous decisions were made and adjust the decisions accordingly. We plan to investigate such dynamic models in both “rolling” and “folding” horizon frameworks (in the rolling horizon framework, each decision epoch covers a fixed number of periods in the future, whereas in the folding horizon framework, we advance towards a given target date and so the decision epochs correspond to an ever-decreasing set of periods).

## Appendix

### Proof of Lemma 2

The decision version of the constrained shortest-path problem for the FCDS graphs defined in this paper can be stated as follows: Given positive numbers  $R$  and  $C$ , determine whether there is an  $(O-D)$ -path in the FCDS graph with  $d$ -length  $\leq R$  and  $c$ -length  $\leq C$ . This decision problem is clearly in NP because the length of a path in a graph can be determined in polynomial time. We next prove that this decision problem is NP-complete by showing that any instance of the partition problem known to be NP-complete

**Table 4.**  $\rho$  values with 5% increase of the violated temporal budgets constraints.

Value	0	0.004	0.009	0.013	0.018	0.022	0.027	0.031	0.036
Frequency	303	135	114	113	68	49	47	39	22
Value	0.040	0.045	0.049	0.054	0.058	0.062	0.067	0.071	more
Frequency	16	10	11	11	8	10	8	6	9

**Table 5.**  $\rho$  values with 5% percent increase of violated temporal budget constraints.

Count	Average	STD	Max	Min
979	0.0135	0.0356	1	0

(see Garey and Johnson 1979) can be reduced in polynomial time to a constrained shortest-path problem on an FCDS graph. Our proof modifies arguments of Wang and Crowcroft (1996).

The data for an instance of the partition problem consists of a set of  $n > 1$  positive integers  $\{a_1, a_2, \dots, a_n\}$ . With  $S = \sum_{i=1}^n a_i$ , the problem is then to determine whether there exists a subset  $I \subseteq \mathcal{N} \equiv \{1, \dots, n\}$  such that  $\sum_{i \in I} a_i = S/2$ . We next construct an instance of the CM development problem such that the given instance of the partition problem has a solution if and only if the corresponding FCDS graph has an  $(O-D)$ -path of  $d$ -length  $\leq S/2 + 1$  and  $c$ -length  $\leq S/2$ .

Without loss of generality, assume that  $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq 1$  (the  $a_i$ s can be ordered using  $O(n \lg n)$  comparisons). Also assume that  $S$  is even and  $a_1 > 1$  (else, the given instance of the partition problem is trivial), implying that  $S > 2$ . Let  $\gamma \equiv a_1/S < 1 < S$ . The CM development problem that we construct has  $\mathcal{M} = \{1, \dots, 2n-1\}$ ,  $\mathcal{W} = \{1\}$ ,  $s_1 = 0$ , and  $\mathcal{K} = \{1\}$ . Hence, the dependence of  $t_m^k$ ,  $d_m^k$ , and  $c_m^k$  on  $k$  can be suppressed, and these are given by

$$(t_m, d_m, c_m) = \begin{cases} \left( \sum_{u=1}^{i-1} S^u + \gamma, \frac{a_{i+1}}{S(S^i - \gamma)}, a_i \right) & \text{if } m = 2i - 1 \text{ for } i \in \{1, \dots, n-1\} \\ \left( \sum_{u=1}^i S^u, \frac{a_{i+1}}{S^{i+1}}, 0 \right) & \text{if } m = 2i \text{ for } i \in \{1, \dots, n-1\} \\ \left( \sum_{u=1}^{n-1} S^u + \gamma, 0, a_n \right) & \text{if } m = 2n - 1 \end{cases} \quad (35)$$

(where  $S^u$  stands for the  $u$ -power of  $S$ ). Also,  $d_0 = a_1/S$  and  $T = \sum_{u=1}^n S^u$ .

Clearly,  $0 < t_1 < \dots < t_{2n-1} < T$ . Also, for  $i = 1, \dots, n-1$ ,  $d_{2i-1} = a_{i+1}/S(S^i - \gamma) > a_{i+1}/S^{i+1} = d_{2i}$  and  $S(S^i - \gamma) - S^i = S^i(S - 1) - a_1 > 0$ ,  $d_{2(i-1)} = a_i/S^i > a_{i+1}/S(S^i - \gamma) = d_{2i-1}$ . Thus,  $d_0 > d_1 > \dots > d_{2n-2} > 0 = d_{2n-1}$ . Following (3), the corresponding FCDS graph is a complete directed graph, i.e., with  $O$  and  $D$  associated with 0 and  $2n$ , respectively, its vertex set is  $V \equiv \{0, 1, \dots, 2n\}$  and its edge set is  $\{(m, m') \in V \times V : m < m'\}$ . Further, for each edge  $e = (m, m')$  for which  $\{v \in V : m < v < m'\}$  contains an even integer, say  $l$ ,  $c_{(m, m')} = c_{m'} = 0 + c_{m'} = c_{(m, l)} + c_{(l, m')}$  and, by Lemma 1,  $d_{(m, m')} > d_{(m, l)} + d_{(l, m')}$ . Consequently, such edges need not be considered in exploring the existence of an  $(O-D)$ -path of  $c$ -length  $\leq S/2$  and  $d$ -length  $\leq S/2 + 1$ . Therefore, one can restrict attention to paths that contain all of the even vertices; such paths are determined by the set of odd vertices that they contain. An illustration of an FCDS graph without the unnecessary edges is given in Figure A.1.

Consider an  $(O-D)$ -path  $\sigma$  that contains all even vertices and whose odd vertices consist of  $\{2i-1 : i \in I\}$  where  $I \subseteq \mathcal{N}$ . Then  $c(\sigma) = \sum_{i \in I} a_i$  and

$$d(\sigma) = \sum_{i \in \mathcal{N} \setminus I} \left( \frac{a_i}{S^i} \right) S^i + \sum_{i \in I} \left( \frac{a_i}{S^i} \right) \gamma$$

$$+ \sum_{i \in I \setminus \{n\}} \left[ \frac{a_{i+1}}{S(S^i - \gamma)} \right] (S^i - \gamma) \\ = \left( S - \sum_{i \in I} a_i \right) + \sum_{i \in I} \left( \frac{a_i}{S^i} \right) \gamma + \sum_{i \in I \setminus \{n\}} \left( \frac{a_{i+1}}{S} \right). \quad (36)$$

In particular,  $c(\sigma) \leq S/2$  if and only if  $\sum_{i \in I} a_i \leq S/2$  and  $d(\sigma) \leq S/2 + 1$  if and only if

$$\frac{S}{2} - 1 + \sum_{i \in I} \left( \frac{a_i}{S^i} \right) \gamma + \sum_{i \in I \setminus \{n\}} \left( \frac{a_{i+1}}{S} \right) \leq \sum_{i \in I} a_i. \quad (37)$$

If  $I = \emptyset$ , then  $d(\sigma) = S > S/2 + 1$ , implying that this case can be excluded. With  $I \neq \emptyset$ ,

$$0 < \sum_{i \in I} \left( \frac{a_i}{S^i} \right) \gamma + \sum_{i \in I \setminus \{n\}} \left( \frac{a_{i+1}}{S} \right) \\ \leq \left( \frac{\sum_{i \in \mathcal{N}} a_i}{S} \right) \gamma + \left( \frac{\sum_{i \in \mathcal{N} \setminus \{1\}} a_i}{S} \right) = \gamma + \frac{S - a_1}{S} = 1. \quad (38)$$

Let  $\Delta \equiv \sum_{i \in I} (a_i/S^i) \gamma + \sum_{i \in I \setminus \{n\}} (a_{i+1}/S)$ . Because the  $a_i$ s and  $S/2$  are integers and  $0 < \Delta \leq 1$  (the latter by (38)),  $S/2 - 1 + \Delta \leq \sum_{i \in I} a_i$  if and only if  $S/2 \leq \sum_{i \in I} a_i$ , i.e., (37) is equivalent to  $S/2 \leq \sum_{i \in I} a_i$ . Therefore,  $c(\sigma) \leq S/2$  together with  $d(\sigma) \leq S/2 + 1$  are equivalent to  $\sum_{i \in I} a_i = S/2$ . Thus, the given instance of the partition problem was reduced to an instance of a decision problem of a constrained shortest path in an FCDS graph.  $\square$

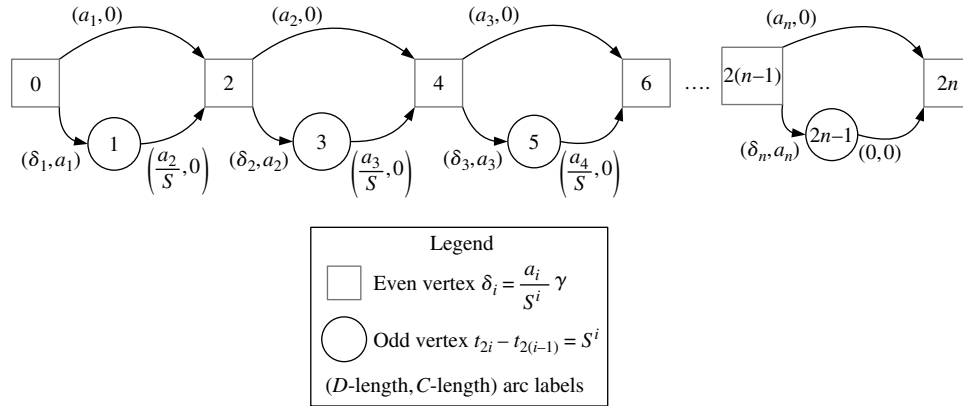
### The General Model Used for the Numerical Experiments

The formulation of the general model we used for the numerical experiments is an extension of the model presented in §4.4 according to §3.3; that is, the time at which the development of each CM starts is a decision variable. Recall that in this model we refer to *plans*, where each plan consists of a pair  $(k, \tau)$ , where  $k$  is the intensity of developing the CM and  $\tau$  is the start time. We let  $\Gamma$  be a finite set of all possible start times. We assume that neither the effectiveness of a CM nor the cost and duration of developing it are affected by  $\tau$ .

As in §3.3, a *policy*  $\pi$  is now defined as a set of triplets  $(m, k, \tau)$  with distinct values of  $m$ , and effective policies are defined in terms of the  $t_m^{(k, \tau)}$  (the completion times). Consider the modification of the multiple-FCDS graph to *multiple-T-FCDS* graph where vertices and edges are defined in terms of the triplets  $(m, k, \tau)$  instead of the pairs  $(m, k)$ . As in §3.3, if the development plan of  $CM_m$  is  $(k, \tau)$ , then during the development period  $[\tau, t_m^{(k, \tau)}]$  a cost  $\hat{c}_m^{(k, \tau)} \equiv c_m^{(k, \tau)} / t_m^{(k, \tau)}$  per unit time is incurred.<sup>3</sup> The total expenditure on  $CM_m$  during the time interval  $I_h = [\underline{T}_h, \bar{T}_h]$  is  $\hat{c}_m^{(k, \tau)} [\min\{\bar{T}_h, t_m^{(k, \tau)}\} - \max\{\underline{T}_h, \tau\}]_+$ . For each edge  $e$  of the multiple-T-FCDS graph and  $h = 1, \dots, H$ , let

$$c_e^h \equiv \begin{cases} \hat{c}_m^{(k, \tau)} [\min\{\bar{T}_h, t_m^{(k, \tau)}\} - \max\{\underline{T}_h, \tau\}]_+ & \text{if } e \text{ terminates at } (m, k, \tau) \\ 0 & \text{if } e \text{ terminates at } D. \end{cases} \quad (39)$$

The total cost associated with effective policy  $\pi$  during the time interval  $I_h$  is then expressed by  $\sum_{e \in E} c_e^h x_e$ , and this sum is subject to the corresponding temporal budget constraint  $C^h$ .

**Figure A.1.** The FCDS graph constructed for the proof of Lemma 2.

Now, consider the total damage associated with an effective policy  $\pi$ . Suppose each weapon  $w$  becomes operational at time  $s_w > 0$ . For each edge  $e$  of the multiple-T-FCDS graph, let  $d_e$  be given by (15), where the  $d_e^w$  values are given by the variant of (14) in which  $k$  is replaced by  $(k, \tau)$ . The total damage associated with policy  $\pi$  is then expressed by  $\sum_{w \in \mathcal{W}} \sum_{e \in E(\sigma_w)} d_e$ . Our formulation is given as follows:

$$\sum_{e \leftarrow (m, k, \tau)^w} x_e = \sum_{e \rightarrow (m, k, \tau)^w} x_e \quad \text{for each } m \in \mathcal{M}, k \in \mathcal{K},$$

$$\tau \in \Gamma \text{ and } w \in \mathcal{W} \quad (40)$$

$$\sum_{e \leftarrow O^w} x_e = 1 = \sum_{e \rightarrow D^w} x_e \quad \text{for each } w \in \mathcal{W}$$

$$x_e \in \{0, 1\} \quad \text{for each } e \in E.$$

In particular, if configuration  $\sigma = \{\sigma_w : w \in \mathcal{W}\}$  corresponds to  $x$  satisfying (40), then

$$\sum_{w \in \mathcal{W}} \sum_{e \in E(\sigma_w)} d_e = \sum_{e \in E} d_e x_e. \quad (41)$$

Given such a solution  $x$ , let  $y(x) = [y(x)_m^{(k, \tau)}]_{(m, k, \tau) \in U}$  be defined by

$$y(x)_m^{(k, \tau)} = \max_w \sum_{e \rightarrow (m, k, \tau)^w} x_e \quad \text{for each } (m, k, \tau) \in U$$

(where  $U$  is redefined to accommodate triplets  $(m, k, \tau) \in \mathcal{M} \times \mathcal{K} \times \Gamma$  instead of pairs  $(m, k) \in \mathcal{M} \times \mathcal{K}$ ); equivalently,  $y(x)$  is the unique solution of

$$y(x)_m^{(k, \tau)} \geq \sum_{e \rightarrow (m, k, \tau)^w} x_e \quad \text{for each } (m, k, \tau) \in U$$

$$\text{and } w \in \mathcal{W}, \tau \in \Gamma \quad (42)$$

$$y(x)_m^{(k, \tau)} \leq \sum_{w \in \mathcal{W}} \sum_{e \rightarrow (m, k, \tau)^w} x_e \quad \text{for each } (m, k, \tau) \in U$$

$$y(x)_m^{(k, \tau)} \in \{0, 1\} \quad \text{for each } (m, k, \tau) \in U.$$

Now, the budget constraints are;

$$\sum_{e \in E} c_e^h x_e \leq C^h \quad \text{for } h = 1, \dots, H \quad (43)$$

and

$$\sum_{(m, k) \in U} c_m^{(k, \tau)} y_m^{(k, \tau)} \leq C. \quad (44)$$

The problem we solved in our numerical experiments is:

$$\min_{x, y} \sum_{e \in E} d_e x_e$$

$$\text{s.t. } (x, y) \text{ satisfies (40), (42), (43), and (44).} \quad (45)$$

## Endnotes

1. The index sets of the intensities of the different CMs are the same only for notational convenience.
2. For example, it is possible to have a policy  $\pi$  that is not  $w$ -effective for any  $w \in \mathcal{W}$ —although  $\pi$  determines a configuration, it is not in the image of  $\text{policy}(\cdot)$ . Also, given a configuration  $\sigma$  and  $w \neq w'$ , it is possible for  $\sigma_w$  to contain a vertex  $(m, k)$  that is not  $w'$ -dominated by any vertex of  $\sigma_{w'}$ —this vertex will then appear in  $\text{path}[\text{policy}(\sigma)]_{w'}$ .
3. In our experiments, we took  $t_m^{(k, \tau)} = t_m^k$  and  $c_m^{(k, \tau)} = c_m^k \forall \tau$ , i.e., the cost rate  $\hat{c}_m^{(k, \tau)}$  holds during the development period, but its value does not depend on the start time  $\tau$ .

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